Group Theory 3

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Lemma:- If $k, l \geq 0$ then

$$(a\,b)(a\,c_1 \cdots c_k\,b\,d_1 \cdots d_l) = (a\,c_1 \cdots c_k)(b\,d_1 \cdots d_l)$$

and

$$(a\,b)(a\,c_1 \cdots c_k)(b\,d_1 \cdots d_l) = (a\,c_1 \cdots c_k\,b\,d_1 \cdots d_l)$$

Proof:- $a \to c_1 \to c_1$, $c_i \to c_{i+1} \to c_{i+1}$ $\forall\, i < k$, $c_k \to b \to a$,

$b \to d_1 \to d_1$, $d_j \to d_{j+1} \to d_{j+1}$ if $j < l$, $d_l \to a \to b$

Similarly the next equality

Def:- If $\alpha \in S_n$ and $\alpha = \beta_1 \cdots \beta_t$ is a complete factorization into disjoint cycles, then parity is defined as, $\prod_{i=1}^{t} \left((-1)^{k_i - 1}\right)$ where $\beta_i = (\beta_{i1}, \beta_{i2}, \ldots, \beta_{ik_i})$

Idea

S_3

odd $(1\,2\,3) \to +1$

odd $(1)(2)(3) \to +1$

even $(1\,2)(1\,3) = (1\,3\,2) \to +1$

$(a_1 a_2)$

$a_2 - a_1$
$+ a_2 - a_1 - 1$

$(a_1 a_2 a_3)$

$a_1\, a_2\, a_3\, a_1$

$\beta_1 \to (\beta_{11}\,\beta_{12} \cdots \beta_{1k})$

$(-1)^{k-1}$

$\beta_1 \quad \beta_2 \quad \beta_3$

$(1\;6)$ $\quad (1\;2\;3) = (1\,3)(1\,2)$

$+1$

$1\;2\;3\;4\;5\;6 \Rightarrow 3\;1\;2\;4\;5\;6$

$6\;2\;3\;4\;5\;1\;6 \quad 5 \quad 4 \quad = 9$

Def:- If $\alpha \in S_n$ and $\alpha = \beta_1 \cdots \beta_t$ is a complete factorization into disjoint cycles, then signum α is defined as $\operatorname{sgn}(\alpha) = (-1)^{n-t}$

Lemma:- If $\beta \in S_n$ and τ is a transposition, then $\operatorname{sgn}(\tau\beta) = -\operatorname{sgn}(\beta)$

Proof:- $\tau = (a\,b)$ $\quad \beta = \beta_1 \beta_2 \cdots \beta_t \to$ disjoint cycles complete factn

If $a, b \notin$ elements of β_1 then $\Rightarrow$ it is done

Proof:- $\tau = (a\ b)$

If $a, b \notin$ elements of β_i then $\Rightarrow$ it is done

If a or $b \in$ elements of β_i,

$a, b \in$ some β_k, $(a\ b)\ \beta_1 \beta_2 \cdots \beta_k \cdots \beta_t$

$\Rightarrow (a\ b)\ \beta_k\ \beta_1 \beta_2 \cdots \beta_{k-1} \beta_{k+1} \cdots \beta_t$

$\Rightarrow (a\ b)\ (a\ c_1 c_2 \cdots c_j\ b\ d_1 d_2 \cdots d_m)\ \beta_1 \beta_2 \cdots \beta_t$

$\Rightarrow \underbrace{(a\ c_1\ c_2 \cdots c_j)(b\ d_1 d_2 \cdots d_m)\ \beta_1 \beta_2 \cdots \beta_t}_{t+1 \text{ terms}} \Rightarrow$ it is done

$a, b \in$ different β_k's, $a \in \beta_{k_1}$, $b \in \beta_{k_2}$,

$(a\ b)\ \beta_1 \cdots \beta_{k_1}\ \beta_{k_2} \cdots \beta_t$

$\Rightarrow (a\ b)\ \beta_{k_1} \beta_{k_2}\ \beta_1 \cdots \beta_t \Rightarrow (a\ b)\ (a\ c_1 c_2 \cdots c_j)\ (b\ d_1 d_2 \cdots d_m)\ \beta_1 \cdots \beta_t$

$\Rightarrow \underbrace{(a c_1 \cdots c_j\ b\ d_1 \cdots d_m)\ \beta_1 \cdots \beta_t}_{t-1 \text{ terms}} \Rightarrow$ it is done

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Theorem - For all $\alpha, \beta \in S_n$,

$$\mathrm{sgn}(\alpha\beta) = \mathrm{sgn}(\alpha)\,\mathrm{sgn}(\beta)$$

Proof:- $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ | $\beta = \beta_1 \beta_2 \cdots \beta_m$

can be in n transpositions | can be m transposition

For $n = 1$, $\alpha = \alpha_1$, by above lemma $\mathrm{sgn}(\alpha\beta) = \mathrm{sgn}(\alpha_1\beta) = -\mathrm{sgn}(\beta) = \mathrm{sgn}(\alpha)\,\mathrm{sgn}(\beta)$

For $n = m$ let it be true

For $n = m+1$, $\mathrm{sgn}(\alpha_1(\alpha_2 \cdots \alpha_m \alpha_{m+1} \beta)) = (\mathrm{sgn}\,\alpha_1)\ \mathrm{sgn}(\alpha_2 \cdots \alpha_{m+1}\beta)$

$= \mathrm{sgn}(\alpha_1 \alpha_2 \cdots \alpha_{m+1})\ \mathrm{sgn}(\beta)$

$= \mathrm{sgn}(\alpha)\ \mathrm{sgn}(\beta)$

Theorem:- A permutation $\alpha \in S_n$ is even if and only if $\mathrm{sgn}(\alpha) = 1$

" " $\alpha \in S_n$ is odd if and only if $\mathrm{sgn}(\alpha) = -1$

Proof :- $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n \rightarrow$ transpositions

$sgn(\alpha) = sgn(\alpha_1)\, sgn(\alpha_2) \cdots sgn(\alpha_n) = (-1)^n$

If $sgn(\alpha) = 1 \Rightarrow sgn(\alpha) = 1 \Rightarrow n$ is even $\Rightarrow$ even permutation

Conversely, if even permutation $n = 2m$, $\Rightarrow sgn(\alpha) = (-1)^{2m} = 1$

Similarly for odd

Q) Show that an r-cycle is an even permutation if and only if r is odd.

Ans :- $(1\ 2\ 3\ \cdots\ r) = \underbrace{(1\, r)(1\, r\!-\!1) \cdots (1\, 2)}_{r-1 \text{ terms}}$

Parity $= (-1)^{r-1}$ if r is odd $\Rightarrow (-1)^{r-1} = 1 \Rightarrow$ even permutation

Q) $\alpha, \beta \in S_n$. If α and β have the same parity then $\alpha\beta$ is even, if α and β have distinct parity then $\alpha\beta$ is odd. Show this

Ans :- $sgn(\alpha\beta) = sgn(\alpha)\, sgn(\beta) = (+1)(+1)$ or $(-1)(-1) = 1$

$sgn(\alpha\beta) = sgn(\alpha)\, sgn(\beta) = (-1)(+1)$ or $(+1)(-1) = -1$

Q) Show that S_n has the same number of even permutations as of odd permutations

Ans :- Let's define a map, $f : S_n \rightarrow S_n$ $\quad a, b \in \{1, 2, \cdots n\}$

$\alpha \rightarrow (a\, b)\alpha$

$sgn(\alpha) = -sgn((a, b)\,\alpha)$

different parity

~~even~~ $\rightarrow$ odd

odd $\rightarrow$ even

$f(\alpha_1) = f(\alpha_2)$

$(a\, b)\alpha_1 = (a\, b)\alpha_2$

$\alpha_1 = \alpha_2$

So this is one-one and as cardinality of domain & codomain is same it is onto. $\Rightarrow$ bijection

$\rightarrow |\text{even}| = |\text{odd}|$